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Chapter 3: Grade Four Areas of Emphasis

By the end of grade four, students understand large numbers and addition, subtraction, multiplication, and division of whole numbers. They describe and compare simple fractions and decimals. They understand the properties of, and the relationships between, plane geometric figures. They collect, represent, and analyze data to answer questions.

Number Sense

1.0 **1.1** **1.2** **1.3** **1.4** 1.5 1.6 1.7 **1.8** **1.9**

2.0 2.1 2.2

3.0 **3.1** **3.2** **3.3** **3.4**

4.0 4.1 **4.2**

Algebra and Functions

1.0 1.1 **1.2** **1.3** 1.4 **1.5**

2.0 **2.1** **2.2**

Measurement and Geometry

1.0 1.1 1.2 1.3 1.4

2.0 **2.1** **2.2** **2.3**

3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 **1.1**

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3

2.0 2.1 2.2 **1.2**

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Mathematical Reasoning

1.0 1.1 1.2

2.0 2.1 2.2 2.3 2.4 2.5 2.6

3.0 3.1 3.2 3.3

Chapter 3: Grade Four

Key Standards

NUMBER SENSE

The Number Sense strand for the fourth grades extends students' knowledge of numbers to both bigger numbers (millions) and smaller numbers (two decimal places).

Up to this point, students have been asked to learn to round numbers to the nearest tens, hundreds, and thousands, probably without knowing why. It is now finally possible to explain why rounding is much more than a mechanical exercise and is in fact an essential skill in the application of mathematics to understanding the world around us. One can use the population figure of the United States for this purpose. According to latest census (year 2000), there are 281,421,906 people in this country. Explain to students that, in either daily conversation or strategic planning, it would be more sensible to use the round-off figure of 280 million rather than the precise figure of 281,421,906 (due to the built-in errors of a project of this size, the impossibility of correctly counting all the people in transit, the impossibility of reaching all homeless people, the difficulty of obtaining total participation, etc.). Therefore, rounding to the nearest ten million in this case becomes a matter of necessity in discarding unreliable and nonessential information.

Standard 1.5 brings out two facts about fractions that are fundamental for students' understanding of this topic: different interpretations of a fraction and the equivalence of fractions. We will discuss them one at a time.

The fact that a fraction such as 3/5 is not only 3 parts of a whole when the whole (the unit) is divided into 5 equal parts but also one part of 3 when 3 is divided into 5 equal parts is so basic that often one uses it without being aware of it. For example, if we are asked in a daily conversation how long one of the pieces of a 3-foot rod is

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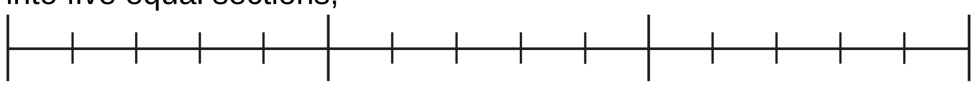
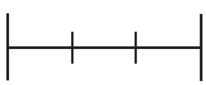
when it is cut into 5 pieces of equal length, we would say without thinking that it is 3/5 of a foot. In so doing we are using the second (division) interpretation of 3/5. On the other hand, it is important to remember that, *according to the part-whole definition of a fraction,* 3/5 of a foot is the length of 3 of the pieces when a 1-foot rod is divided into 5 pieces of equal length. *Students need an explanation of why these two lengths are equal.* One way to explain is to divide each foot of the 3-foot rod

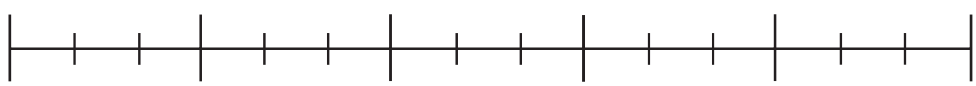
into five equal sections,



Each section is the result of dividing 1 foot into 5 equal parts, and so by the part-whole definition of a fraction, the length of three such sections joined together



is 3/5 of a foot. But we can clearly group the 15 (=3 x 5) sections of the 3-foot rod to divide the rod into five equal lengths,



and we see that 3/5 of a foot is identical to the length of one of the pieces when a 3-foot rod is divided into 5 equal lengths. Therefore the part-whole and division definitions of a fraction coincide.

This explanation continues to be valid when the fraction 3/5 is replaced by any other fraction $\frac{a}{b}$. The concept of the equivalence of fractions lies at the core of almost every mathematical consideration related to fractions. Students should be given every opportunity to understand why $2/5 = 14/35$, why $5/4 = 40/32$, or why $\frac{a}{b} = \frac{na}{nb}$ for any whole number *a, b, n* (it will always be understood that $b \neq 0$ and $n \neq 0$). One can

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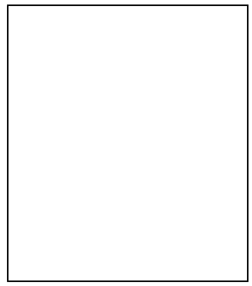
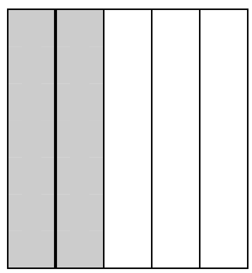
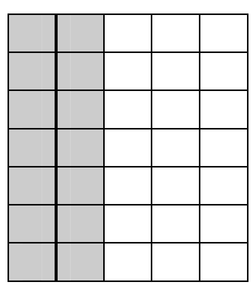
explain 2/5 = 14/35 by use of a picture, provided the context of the picture is carefully laid out. Let then the unit 1 be fixed as the *area* of the unit square:



The fraction 2/5 is then 2 parts of the unit square when it is divided into 5 parts of equal area. We do the equi-division vertically:



Since each vertical strip represents 1/5, the shaded region represents 2/5. The fraction 14/35 is, on the other hand, 14 parts of the unit square when it is divided into 35 parts of equal area. We can achieve the desired equi-division into 35 parts by adding 7 equally spaced horizontal divisions of the unit square to the preceding vertical division:



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Now the unit square is divided into 35 small rectangles of the same size, so each small rectangle is 1/35. Since there are 14 of these small rectangles in the shaded region, the shaded region therefore represents not only 2/5, but also 14/35.

The preceding reasoning is perfectly general, but for fourth graders, generality should be soft-pedaled. Mentioning $\frac{a}{b} = \frac{na}{nb}$ in passing may be enough. What needs special emphasis, however, is the immediate consequence of the equivalence of $\frac{a}{b}$ and $\frac{na}{nb}$, namely, the fact that *any two fractions can be written as two fractions with the same denominator.* Thus if $\frac{a}{b}$ and $\frac{c}{d}$ are two given fractions, they can be rewritten as $\frac{ad}{bd}$ and $\frac{bc}{bd}$, which have the same denominator *bd.* This fact has enormous implications when we come to the addition of fractions.

The consideration of why a fraction has a division interpretation, as given above, also sheds light on the teaching of Standard 1.7. To represent the fraction 3/5 as a decimal, for example, we divide the given unit into 10 equal parts. This is best represented on the number line as 9 equi-distant markings of the line segment from 0 to 1. By taking the 2nd, 4th, 6th, and 8th markings, we obtain a division of the unit into 5 equal parts. Since the fraction 3/5 is 3 of these parts, it is the 6th marking. But the 10 markings represent 0.1, 0.2, . . . 0.9, and therefore the 6th marking is 0.6. This shows 3/5 is 0.6.

The next standards are basic and new standards:

1.8 Use concepts of negative numbers (e.g., on a number line, in counting, in temperature, in “owing”).

1.9 Identify on the number line the relative position of positive fractions, positive mixed numbers, and positive decimals to two decimal places.

These standards can be difficult for students to learn if the required background material—ordering of whole numbers and comparison of fractions and decimals—is not presented carefully. The importance of these standards requires that close

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attention be paid to assessment. The second standard is about “simple” decimals, that is, decimals up to two decimal places. We have not discussed decimals up to this point, but it is time to note that, for decimals up to two decimal places, their addition and subtraction can be completely modeled by money and can therefore be done informally. Looking ahead, though, to when the arithmetic operations of (finite or terminating) decimals of any number of decimal digits are taken up in grade five, we see it is imperative to inform students that, formally, a finite decimal is a fraction whose denominator is a power of 10. This awareness is important in the teaching of decimals in grade four. (To develop this awareness, it is helpful to describe decimals such as 1.03 verbally as one and three-hundredths, not as one point oh three).

The third topic in the Number Sense strand is also especially important. This and its four substandards all involve the use of the standard algorithms for addition, subtraction, and multiplication of multidigit numbers as well as the standard algorithm for division of a multidigit number by a one-digit number. As with simple arithmetic, mastery of these skills will require extensive practice over several grade levels, as described in Chapter 4, “Instructional Strategies.” The emphasis on Standard 3.1 is, however, on a formal (mathematical) understanding of the addition and subtraction algorithms for whole numbers. It is important for students to see the prominent role played by the commutative law and especially the associative law of addition in the explanation of these algorithms. We should note also that students’ prior familiarity with the skill component of these algorithms is essential for their understanding here, for the following reason. If they are shaky in the mechanics of these algorithms, their minds would be preoccupied with the mechanics and would not be free to appreciate the reasoning behind the mechanics.

Standard 3.2 is about the reasoning that supports the multiplication and division algorithms at least in simple situations (two-digit multipliers and one-digit divisors). It is a bit awkward here because the key fact is the distributive law, which is not

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mentioned until grade five (Algebra and Functions, Standard 1.3). However, if care and patience are conjoined, students can learn the distributive law. For the division algorithm, there is a new element, namely, division-with-remainder: if a and b are whole numbers, then there are always whole numbers q and r so that $a = qb + r$, where r is a whole number strictly smaller than the divisor b. The division algorithm can then be explained as an iterated application of this division-with-remainder.

The fourth topic, “students know how to factor small whole numbers,” is needed for the discussion of the equivalence of fractions. It also includes the requirement that students understand what a prime number is. The concept of primality is important yet often difficult for students to understand fully. Students should also know the prime numbers up to 50. For these reasons the preparation for the discussion of prime numbers should begin no later than the third grade. Students who understand prime numbers will find it easier to understand the equivalence of fractions and to multiply and divide fractions in grades five, six, and seven.

ALGEBRA AND FUNCTIONS

In the fourth grade the Algebra and Functions strand continues to grow in importance. All five of the subtopics under the first standard are important. But the degree to which students need to understand these strands differs. The following standards *do not need undue emphasis*:

1.2 Interpret and evaluate mathematical expressions that now use parentheses.

1.3 Use parentheses to indicate which operation to perform first when writing expressions containing more than two terms and different operations.

These standards involve nothing more than notation. The real skill is learning how to write expressions unambiguously so that others can understand them. However, it would be appropriate at this point to explain carefully to students why the associative

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and commutative laws are significant and why arbitrary sums or products, such as $115 + 6 + (-6) + 4792$ or $113 \times 212 \times 31 \times 11$, do not have to be ordered in any particular way, nor do they have to be calculated in any particular order.

Standards 1.4 and 1.5, which relate to functional relationships, are much more important theoretically. In particular, students should understand Standard 1.5 because it takes the mystery out of the topic.

1.5 Understand that an equation such as $y = 3x + 5$ is a prescription for determining a second number when a first number is given.

One way to understand an equation such as $y = 3x + 5$ is to work through many pairs of numbers (x, y) to see if they satisfy this equation. For example, (1, 8) and (0, 5) do, but (–1, 3) and (2, 10) do not.

The second algebra standard is, however, basic:

2.0 Students know how to manipulate equations.

This standard and the two basic rules that follow, if understood now, will clarify much of what happens in mathematics and other subjects from the fifth grade through high school.

2.1 Know and understand that equals added to equals are equal.

$2 + 1 = 3$, and $7 - 2 = 5$, so therefore $2 + 1 + 5 = 3 + 7 - 2$

2.2 Know and understand that equals multiplied by equals are equal.

$2 + 1 = 3$, and $4 \times 5 = 20$, so therefore $(2 + 1) \times 20 = 3 \times (4 \times 5)$

However, if these concepts are not clear, difficulties in later grades are virtually guaranteed. Therefore, careful assessment of students' understanding of these principles should be done here.

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MEASUREMENT AND GEOMETRY

The Measurement and Geometry strand for the fourth grade contains a few key standards that students will need to understand completely. The first standard (1.0) relates to perimeter and area. The students need to understand that the area of a rectangle is obtained by multiplying length by width and that the perimeter is given by a linear measurement. The intent of most of this standard is that students know the reasons behind the formulas for the perimeter and area of a rectangle and that they can see how these formulas work when the perimeter and area vary as the rectangles vary.

A more basic standard is the second one:

2.0 Students use two-dimensional coordinate grids to represent points and graph lines and simple figures.

Although the material in this standard is basic and is not presented in depth, this concept must be presented carefully. Again, students who are confused at this point will very likely have serious difficulties in the later grades—not just in mathematics, but in the sciences and other areas as well. Therefore, careful assessment is necessary. We call special attention to the need of students to understand the graphs of the equations $x = c$ and $y = c$ for a constant *c.* These are what are commonly called vertical and horizontal lines, respectively. What has to be done is to get hold of some points on these graphs strictly according to the definition of the graph of an equation as the set of all points (*x, y,*) whose coordinates satisfy the given equation. Unless this is painstakingly done, these graphs will continue to be nothing but magic through the rest of students' schooling.

In connection with Standard 3.0, teachers should introduce the symbol $\perp$ for perpendicularity. Incidentally, this is the time to introduce the abbreviated notation *ab* in place of the cumbersome $a \times b$.

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Elaboration

Knowledge of multiplication and division facts should be reassessed at the beginning of the school year, and systematic instruction and practice should be provided to enable students to reach high degrees of automaticity in recalling these facts. This process is described for addition in grade two (see “Elaboration”).

Reading and writing thousands and millions numbers with one or more zeros in the middle can be particularly troublesome for students (Seron and Fayol 1994). Therefore, assessment and teaching should be thorough so that students are able to read and write difficult numbers, such as 300,200 and 320,000. Students need to understand that zeros in different positions represent different place values—tens, hundreds, thousands, and so forth—and they need practice in working with these types of numbers (e.g., determining which is larger, 320,000 or 300,200, and translating a verbal label, “one million two hundred thousand,” into the Arabic numeral representation, 1,200,000).

To be able to apply mathematics in the real world, to understand the way in which numbers distribute on the number line, and ultimately to study more advanced topics in mathematics, students need to understand the concept of “closeness” for numbers. It is probably not wise to push too hard on the notion of “close enough” while students are still struggling with the abstract idea of a number itself. However, by now they should be ready for this next step. A discussion of rounding should emphasize that one rounds off only if the result of rounding is “close enough.”

Students need to understand fraction equivalencies related to the ordering and comparison of decimals. Students must understand, for instance, that 2/10 = 20/100, then equate those fractions to decimals.

The teaching of the conversion of proper and improper fractions to decimals should be structured so that students see relationships (e.g., the fraction 7/4 can be

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converted to $4/4 + 3/4$, which in turn equals 1 and 3/4). The fourth grade standards do not require any arithmetic with fractions; however, practice with addition and subtraction of fractions (converting to like denominators) must be continued in this grade because these concepts are important in the fifth grade. Students can also be introduced to the concept of unlike denominators in preparation for the following year. Building students' skills in finding equivalent fractions is also important at this grade level.

The standards require that students know the definition of prime numbers and know that many whole numbers decompose into products of smaller numbers in different ways. Using the number 150 as an example, they should realize that $150 = 5 \times 30$ and $30 = 5 \times 6$; therefore, $150 = 5 \times 5 \times 6$, which can be decomposed to $5 \times 5 \times 3 \times 2$. Students will be using these factoring skills extensively in the later grades. Even though determining the prime factors of all numbers through 50 is a fifth grade standard, practice on finding prime factors can begin in the fourth grade. Students should be given extensive practice over an extended period of time with finding prime factors so that they can develop automaticity in the factoring process (see Chapter 4, "Instructional Strategies"). By the end of the fifth grade, students should be able to determine with relative ease whether any of the prime numbers 2, 3, 5, 7, or 11 are factors of a number less than 200.

Multiplication and division problems with multidigit numbers are expanded. Division problems with a zero in the quotient (e.g., $4233/6 = 705.5$) can be particularly difficult for students to understand and require systematic instruction.

The Number Sense Standards 3.1 and 3.2 call for "understanding of the standard algorithm" (see the glossary). To present this concept, the teacher sketches the reasons why the algorithm works and carefully shows the students how to use it. (Any such explanation of the multiplication and division algorithms would help students to deepen their understanding and appreciation of the distributive law.) The

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students are not expected to reproduce this discussion in any detail, but they are expected to have a general idea of why the algorithm works and be able to expand it in detail for small numbers.

As the students grow older, this experience should lead to increased confidence in understanding these *and similar* algorithms, knowledge of how to construct them in other situations, and the importance of verifying their correctness before relying on them. For example, the process of writing any kind of program for a computer begins with creating algorithms for automating a task and then implementing them on the machine. Without hands-on experience like that described above, students will be ill-equipped to construct correct programs.

Considerations for Grade-Level Accomplishments in Grade Four

The most important mathematical skills and concepts for students in grade four to acquire are described as follows:

- Multiplication and division facts. Students who enter the fourth grade without multiplication facts committed to memory are at risk of having difficulty as more complex mathematics is taught. Students' knowledge of basic facts needs to be assessed at the beginning of the school year. Systematic daily practice with multiplication and division facts needs to be provided for students who have not yet learned them.
- Addition and subtraction. Mentally adding a two-digit number and a one-digit number is a component skill for working multiplication problems that was targeted in the second grade. Students have to add the carried number to the product of two factors (e.g., 34×3). Students should be assessed on the ability to add numbers mentally (e.g., $36 + 7$) at the beginning of the school year, and

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systematic practice should be provided for students not able to work the addition problems mentally.

- Reading and writing numbers. Reading and writing numbers in the thousands and millions with one or more zeros in the middle can be particularly troublesome for students. Assessment at the beginning of the fourth grade should test students on reading and writing the more difficult thousand numbers, such as 4,002 and 4,020. When teaching students to read 5- and 6-digit numbers, teachers should be thorough so that students can read, write, and distinguish difficult numbers, such as 300,200 and 320,000.
- Fractions equal to one. Understanding fractions equal to one (e.g., 8/8 or 4/4) is important for understanding the procedure for working with equivalent fractions. Students should have an in-depth understanding of how to construct a fraction that equals one to suit the needs of the problem; for example, should a fraction be 32/32 or 17/17? When the class is working on equivalent fraction problems, the teacher should prompt the students on how to find the equivalent fraction or the missing number in the equivalent fraction. The students find the fraction of one that they can use to multiply or divide by to determine the equivalent fraction. (This material is discussed in depth in Appendix A, “Sample Instructional Profile.”)
- Multiplication and division problems. Multiplication problems in which either factor has a zero are likely to cause difficulties. Teachers should provide extra practice on problems such as 20×315 and 24×308. Division problems with a zero in the answer may be difficult for students (e.g., 152/3 and 5115/5). Students will need prompting on how to determine whether they have completed the problem of placing enough digits in the answer. (Students who consistently find problems with zeros in the answer difficult to solve may also have difficulties with the concept of place value. Help should be provided to remedy this situation quickly.)

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- Order of operations. In the fourth grade students start to handle problems which freely mix the four arithmetic operators, and in this grade the issue of order of operation needs to be addressed explicitly. Students need to know already the convention of order of operations, the precedence of multiplication and division over addition and subtraction, and the implied left-to-right order of evaluation. Parentheses introduce a new way to modify that convention, and Algebra and Functions (AF) Standard 1.2 addresses this explicitly.

  This is also the time to expose the student to the *convenience* of this convention. Students are already taught that an equation is a prescription to determine a second number when a first number is given (AF 1.5) in problem situations and in number sentences, and the clarity of $5x + 3$ over $(5x) + 3$ can be easily demonstrated. This is also the proper time to start weaning students out of the explicit notation of the multiplication symbol, comparing expressions such as $5 \times A + 3$ or $5 \cdot A + 3$ with $5A + 3$.

  Finally, in grade six, the topic of order of operations essentially comes to its completion. A comparison should be done between the associativity of addition and multiplication versus the non-associativity of subtraction and division. A demonstration should be given of how replacement of subtraction by the equivalent addition of negative numbers, or multiplication with a reciprocal instead of division, solves the associativity problem. In other words the non-associativity of the sentence:

$$(9 - 4) - 2 \neq 9 - (4 - 2)$$

  should be compared with the restored associativity when we replace subtraction by addition of the negative value:

$$[9 + (-4)] + (-2) = 9 + [(-4) + (-2)]$$

  In a similar fashion, although we have no associativity with division,

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$$(18 \div 2) \div 3 \neq 18 \div (2 \div 3),$$

when we replace the division with the multiplication by a reciprocal, the associativity returns:

$$(18 \cdot {}^{1}\!/_{2}) \cdot {}^{1}\!/_{3} = 18 \cdot ({}^{1}\!/_{2} \cdot {}^{1}\!/_{3})$$

Now, finally, the student can be exposed to the complete reasoning behind the convention of order of operations. The awkward replacement by the inverse operations, or the need for parentheses, can be much reduced by the application of left-to-right evaluation and the precedence of operators. Would we rather have $(3 \cdot (a^2)) - (5a) + 3$ instead of $3a^2 - 5a + 3$?

However, after all is said and done, we should not forget that mathematical writing also serves to *communicate.* So when an expression is complex and can easily be misinterpreted, it does not hurt to throw in a pair of parentheses here and there, even if they are not strictly required. Students should be encouraged to write $8 - ((12 \div 4) \div 2) \cdot 3 + 3$ instead of $8 - 12 \div 4 \div 2 \cdot 3 + 3$, as the former is less tempting to incorrectly divide 4 by 2 or incorrectly multiply 2 by 3. The use of a horizontal fraction line for division, such as $\frac{a}{b}$ instead of the division symbol $a \div b$, as well as the liberal use of spaces, should also be encouraged to enhance readability and reduce errors. Surely $8 - \dfrac{12 \cdot 3}{4 \cdot 2} + 3$ is even clearer and less error prone than any one of the previous two forms of the same expression.

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Chapter 3: Grade Five Areas of Emphasis

By the end of grade five, students increase their facility with the four basic arithmetic operations applied to fractions and decimals and learn to add and subtract positive and negative numbers. They know and use common measuring units to determine length and area and know and use formulas to determine the volume of simple geometric figures. Students know the concept of angle measurement and use a protractor and compass to solve problems. They use grids, tables, graphs, and charts to record and analyze data.

Number Sense

1.0 1.1 **1.2** 1.3 **1.4** **1.5**

2.0 **2.1** **2.2** **2.3** 2.4 2.5

Algebra and Functions

1.0 1.1 **1.2** 1.3 **1.4** **1.5**

Measurement and Geometry

1.0 **1.1** **1.2** **1.3** 1.4

2.0 **2.1** **2.2** 2.3

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3 **1.4** **1.5**

Mathematical Reasoning

1.0 1.1 1.2

2.0 2.1 2.2 2.3 2.4 2.5 2.6

3.0 3.1 3.2 3.3

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Chapter 3: Grade Five

Key Standards and Elaboration

A significant development in students' mathematics education occurs in grade five. This is the beginning of a three-year sequence (grades 5 through 7) that provides the mathematical foundation of rational numbers. Fractions and decimals have been taught piecemeal up to this point. For example, only decimals with two decimal places are discussed in the fourth grade, and only fractions with the same denominator (or if one denominator is a multiple of the other) are added or subtracted up to grade four. Now both fractions and decimals will be systematically discussed for the next three years. The demand on students' ability to reason goes up ever so slightly at this point, and the teaching of mathematics must correspondingly reflect this increased demand.

By the time students have finished the fourth grade, they should have a basic understanding of whole numbers and some understanding of fractions and decimals. Students at this grade level are expected to have mastered multiplication and division of whole numbers. They should also have had some exposure to negative numbers. These skills will be enhanced in the fifth grade. An important standard focused on enhancing these skills is Number Sense Standard 1.2.

NUMBER SENSE

1.2 Interpret percents as a part of a hundred; find decimal and percent equivalents for common fractions and explain why they represent the same value; compute a given percent of a whole number.

The fact that a fraction *c*/*d* is both "*c* parts of a whole consisting of *d* equal parts" and "the quotient of the number *c* divided by the number *d*" was first mentioned in Number Sense Standard 1.5 of grade four. As discussed earlier in grade four, this

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fact must be *carefully explained* rather than decreed by fiat, as is the practice in most school textbooks. The importance of providing logical explanations for all aspects of the teaching of fractions cannot be overstated because the students’ fear of fractions and the mistakes related to them appear to underlie the failure of mathematics education. Once c/d is clearly understood to be the division of c by d, then the conversion of fractions to decimals can be explained logically.

Students will also continue to learn about the relative positions of numbers on the number line, above all, those of negative whole numbers. Negative whole numbers are especially important because, for the first time, they play a major part in core number-sense expectations. Standard 1.5 is important in this regard.

> **1.5** Identify and represent on a number line decimals, fractions, mixed numbers, and positive and negative integers.

The correct placement of positive fractions on the number line implies that students will need to order and compare fractions. Identifying numbers as points on the real line is an important step in relating students’ concepts of arithmetic to geometry. This fusion of arithmetic and geometry, which is ubiquitous in mathematics, adds a new dimension to students’ understanding of numbers.

Inasmuch as mixed numbers is one of the things that terrorize elementary students, one must approach Standard 1.5 carefully. First, students should not be made to think of “proper” and “improper” fractions as distinct objects; they should be given to understand that these are nothing more than different examples of the same concept—namely, a fraction. Identifying fractions as points on the number line (so that one point is no different from any other point) would go a long way toward eliminating most of this misconception. With that understood, the teacher can now mention that for improper fractions, there is an *alternate representation*. For example, on the number line $^{5}/_{4}$ is beyond 1 by the amount of $^{1}/_{4}$, so $1\frac{1}{4}$ is a reasonable

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alternate notation. Similarly, $^{11}/_3$ is $^{2}/_3$ beyond 3 on the number line, so $3^2/_3$ is also a reasonable alternate notation. When a fraction such as $^{5}/_4$ or $^{11}/_3$ is written as $1\frac{1}{4}$ or $3\frac{2}{3}$, it is said to be a mixed number. In general, fifth graders should be ready for the general explanation of how to write an improper fraction as a mixed number by using division-with remainder: If we suppose $\frac{a}{b}$ is an improper fraction, then we can rewrite it as a mixed number in the following way. The division of the whole number *a* by the whole number *b* is expressed as $a = qb + r$, where *q* is the quotient and the remainder *r* is the whole number strictly less than *b*. Then the fraction $\frac{a}{b}$ is, *by definition*, written as the mixed number $q\frac{r}{b}$. Notice that $\frac{r}{b}$ is a proper fraction. The important point to emphasize is that a mixed number is just a clearly prescribed way of rewriting a fraction, and no fear needs to be associated with it.

But the most important aspect of students' work with negative numbers is to learn the rules for doing the basic operations of arithmetic with them, as represented in the following standard:

2.1 Add, subtract, multiply, and divide with decimals; add with negative integers; subtract positive integers from negative integers; and verify the reasonableness of the results.

In the fifth grade, students learn how to add negative numbers and how to subtract positive numbers from negative numbers. At this point students should find it profitable to interpret these concepts geometrically. Adding a positive number *b* shifts the point on the number line to the right by *b* units, and adding a negative number -*b* shifts the point on the number line to the left by *b* units, and so forth. Multiplication and division of negative numbers should not be taken up in the fifth grade because division by negative numbers leads to negative fractions, which have not yet been introduced. Although Standard 2.1 is listed before Standards 2.3 and 2.4 on the addition and multiplication of fractions, the teaching of decimals must rest on the

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concept of fractions and their arithmetic operations. Formally we define a finite *decimal* as a fraction whose denominator is a power of 10. Without this precise definition, there would be difficulty with an explanation of why the addition and subtraction of decimals are reduced to the addition and subtraction of whole numbers so that the algorithms of the latter can be applied. More to the point, without this precise definition, it would be essentially impossible to explain the rule regarding the decimal point in the multiplication and division of decimals. For example, 2.4×0.37 can be computed by $24 \times 37 = 888$ and since there are three decimal places in both numbers altogether, the usual rule says $2.4 \times 0.37 = 0.888$. The reason, based on the precise definition of a decimal, is that, by definition, $2.4 = 24/10$ and $0.37 = 37/100$ so that $2.4 \times 0.37 = (24/10 \times (37/100) = (24 \times 37) / 1000 = 888/1000 = 0.888$. Many textbooks put the arithmetic operations of decimals ahead of the discussion of fractions, and in general do not bother with a definition of decimals. This creates difficulty for the classroom teacher.

The introduction of the general division algorithm is also important, but it can be complicated and consequently difficult for many students to master. In particular, the skills needed to find the largest product of the divisor with an integer between 0 and 9 that is less than the remainder are likely to be demanding for fifth grade students. Students should become comfortable with the algorithm in carefully selected cases in which the numbers needed at each step are clear. Putting such a problem in context may help. For instance, the students might imagine dividing 153 by 25 as packing 153 students into a fleet of buses for a field trip, with each bus carrying a maximum of 25 passengers. Drawing pictures to help with the reasoning, if necessary, can help students to see that it takes six buses with three students left over; those three students get to enjoy being in the seventh bus with room to spare. But it seems both unnecessary and unwise at this stage to extend the concepts beyond what is presented here. The important standard for students to achieve is:

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2.2 Demonstrate proficiency with division, including division with positive decimals and long division with multidigit divisors.

The most essential number-sense skills that students should learn in the fifth grade are the addition and subtraction of fractions (Standards 2.3) and a little bit of multiplying and dividing fractions (Standards 2.4 and 2.5). At this point of students' mathematics education, it is necessary that they recognize fractions as numbers, which are on the same footing as whole numbers and can therefore be added, multiplied, and so forth. In other words fractions are a special collection of points on the number line that include the whole numbers. To add $a/b+c/d$ for example, we look to the addition of whole numbers for a model. Since 3 + 8 is just the length of the combined segments when a segment of length 3 is concatenated with a segment of length 8, likewise we define a/b + c/d to be the length of the combined segments when a segment of length a/b is concatenated with a segment of length c/d. The computation of this combined length is complicated by the fact that b may be different than d. But the concept of equivalent fractions shows how any two fractions can be made to have the same denominator, namely, $a/b=ad/bd \; and \; c/d=cb/bd$. Therefore, if we think of $1/bd$ as our basic unit, then a/b is ad copies of such a unit and c/d is bc copies of such a unit. Combining them, therefore, shows that $a/b+c/d$ is $ad+bc$ copies of such a unit $1/bd$; that is, $a/b+c/d=\left(ad+bc\right)/bd$:

This is a simple way to obtain a formula for adding fractions. But we should note that this formula is not the definition of adding fractions,which is modeled after the addition of whole numbers. The addition of fractions given should be explained in terms of the least common multiple of the denominators

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Once students have mastered these basic skills with fractions, problems involving concrete applications can be used to provide practice, and to promote students' technical fluency with fractions.

Two main skills are involved in reducing fractions: factoring whole numbers in order to put fractions into reduced forms and understanding the basic arithmetic skills involved in this factoring. The two associated standards that should be emphasized are:

1.4 Determine the prime factors of all numbers through 50 and write the numbers as the product of their prime factors by using exponents to show multiples of a factor (e.g., $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$).

2.3 Solve simple problems, including ones arising in concrete situations involving the addition and subtraction of fractions and mixed numbers (like and unlike denominators of 20 or less), and express answers in the simplest form.

The instructional profile with fractions, which appears later in this chapter, gives many ideas of how to approach this topic. Students may profit from the use of the Sieve of Eratosthenes (see the glossary) in connection with Standard 1.4.

Standard 2.4 asks for the introduction to the multiplication and division of fractions. This is a topic that will be taken up in earnest in grade six, but it is important at this point to remind students of the meaning of division among whole numbers as an *alternate way of writing multiplication*. In other words if $4 \times 7 = 28$, then, *by definition*, we write $28 \div 7 = 4$, or in general, if $a \times b = c$, then we write $c \div b = a$. One can get students used to this idea of "division as a different expression of multiplication" by drills or manipulatives. Once this idea sinks in, they will be ready for the corresponding situation with fractions; that is, if *a, b*, and *c* are fractions, then again by definition, $a \times b = c$ means the same as $c \div b = a$. Using simple fractions, such as $b = \frac{1}{2}$ or $\frac{1}{3}$ and $c = 6$ or 24, and by drawing pictures if necessary, one can easily

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illustrate why $12 \times \frac{1}{2} = 6$ is the same as $6 \div \frac{1}{2} = 12$ or why $24 \times \frac{1}{3} = 8$ is the same as $8 \div \frac{1}{3} = 24$.

ALGEBRA AND FUNCTIONS

The Algebra and Functions strand for grade five presents one of the key steps in abstraction and one of the defining steps in moving from simply learning arithmetic to learning mathematics: the replacement of numbers by variables.

1.2 Use a letter to represent an unknown number; write and evaluate simple algebraic expressions in one variable by substitution.

The importance of this step, which requires *reasoning rather than simple manipulative facility,* mandates particular care in presenting the material. The basic idea that, for example, $3x + 5$ is a shorthand for an infinite number of sums, $3(1) + 5$, $3(2.4) + 5$, $3(11) + 5$, and so forth, must be thoroughly presented and understood by students; and they must practice solving simple algebraic expressions. But it is probably a mistake to push too hard here—teachers should not overdrill. Instead, they should check for students' understanding of concepts, perhaps providing students with some simple puzzle problems to give them practice in writing an equation for an unknown from data in a word problem.

Again, in the Algebra and Functions strand, the following two standards are basic:

1.4 Identify and graph ordered pairs in the four quadrants of the coordinate plane.

1.5 Solve problems involving linear functions with integer values; write the equation; and graph the resulting ordered pairs of integers on a grid.

MEASUREMENT AND GEOMETRY

Finally, in Measurement and Geometry these three standards should be emphasized:

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1.1 Derive and use the formula for the area of a triangle and of a parallelogram by comparing each with the formula for the area of a rectangle (i.e., two of the same triangles make a parallelogram with twice the area; a parallelogram is compared with a rectangle of the same area by cutting and pasting a right triangle on the parallelogram).

2.1 Measure, identify, and draw angles, perpendicular and parallel lines, rectangles, and triangles by using appropriate tools (e.g., straightedge, ruler, compass, protractor, drawing software).

2.2 Know that the sum of the angles of any triangle is 180° and the sum of the angles of any quadrilateral is 360° and use this information to solve problems.

Students need to *commit to memory* the formulas for the area of a triangle, a parallelogram, a rectangle, and the volume of a rectangular solid.

The fact that the angle sum of a triangle is 180° is one of the basic facts of plane geometry, but for students in grade five, it is more important to convince them of this fact through direct measurements than to give a proof.

STATISTICS, DATA ANALYSIS, AND PROBABILITY

The ability to graph functions is an essential fundamental skill, and there is no doubt that linear functions are the most important for applications of mathematics. As a result, the importance of these topics can hardly be overestimated. Closely related to these standards are the following two standards from the Statistics, Data Analysis, and Probability strand:

1.4 Identify ordered pairs of data from a graph and interpret the meaning of the data in terms of the situation depicted by the graph.

1.5 Know how to write ordered pairs correctly; for example, (*x, y*).

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These standards indicate the ways in which the skills involved in the Algebra and Functions strand can be reinforced and applied.

Considerations for Grade-Level Accomplishments in Grade Five

At the beginning of grade five, students need to be assessed carefully on their knowledge of the core content taught in the lower grades, particularly in the following areas:

— Knowledge and fluency of basic fact recall, including addition, subtraction, multiplication, and division facts (By this level, students should know all the basic facts and be able to recall them instantly.)

— Mental addition—The ability to mentally add a single-digit number to a two-digit number

— Rounding off numbers in the hundreds and thousands to the nearest ten, hundred, or thousand and rounding off two-place decimals to the nearest tenth

— Place value—The ability to read and write numbers through the millions

— Knowledge of measurement equivalencies, both customary and metric, for time, length, weight, and liquid capacity

— Knowledge of prime numbers and the ability to determine prime factors of numbers up to 50

— Ability to use algorithms to add and subtract whole numbers, multiply a two-digit number and a multidigit number, and divide a multidigit number by a single-digit number

— Knowledge of customary and metric units and equivalencies for time, length, weight, and capacity

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All of the topics listed previously need to be taught over an extended period of time. A systematic program must be established to enable students to reach high rates of accuracy and fluency with these skills.

Important mathematical skills and concepts for students in grade five to acquire are as follows:

- **Understanding long division.** Long division requires the application of a number of component skills. Students must be able to round tens and hundreds numbers and work estimation problems, divide a two-digit number into a two- or three-digit number mentally and with paper and pencil, and do the steps in the division algorithm. For grade five it suffices to concentrate on problems in which the estimations give the correct numbers in the quotient. This algorithm needs to be taught efficiently so that excessive amounts of instructional time are not required.
- **Adding and subtracting fractions with unlike denominators.** See the instructional profile (Appendix A) on adding and subtracting fractions with unlike denominators.
- **Working with negative numbers.** The standards call for students to add and subtract negative numbers. Students must be totally fluent with these two operations. Students often become confused with operations with negative numbers because too much is introduced at once, and they do not have the opportunity to master one type before another type is introduced. This material must be presented carefully.
- **Ordering fractions and decimal numbers.** Students can use fraction equivalence skills for comparing fractions and for converting fractions to decimals. Students need to know that $3/4 = 75/100 = 0.75 = 75\%$.
- **Working with percents.** To compute a given percent of a number, students can convert the percent to a decimal and then multiply. Students must know that 6%

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translates to 0.06 (percents under ten percent can be troublesome). Students should be assessed on their ability to multiply decimals by whole numbers before work begins on this type of problem.

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Chapter 3: Grade Six Areas of Emphasis

By the end of grade six, students have mastered the four arithmetic operations with whole numbers, positive fractions, positive decimals, and positive and negative integers; they accurately compute and solve problems. They apply their knowledge to statistics and probability. Students understand the concepts of mean, median, and mode of data sets and how to calculate the range. They analyze data and sampling processes for possible bias and misleading conclusions; they use addition and multiplication of fractions routinely to calculate the probabilities for compound events. Students conceptually understand and work with ratios and proportions; they compute percentages (e.g., tax, tips, interest). Students know about *pi* and the formulas for the circumference and area of a circle. They use letters for numbers in formulas involving geometric shapes and in ratios to represent an unknown part of an expression. They solve one-step linear equations.

Number Sense

1.0 **1.1** **1.2** **1.3** **1.4**

2.0 2.1 2.2 **2.3** **2.4**

Algebra and Functions

1.0 **1.1** 1.2 1.3 1.4

2.0 2.1 **2.2** 2.3

3.0 3.1 3.2

Measurement and Geometry

1.0 **1.1** 1.2 1.3

2.0 2.1 **2.2** 2.3

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Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3 1.4

2.0 2.1 **2.2 2.3 2.4 2.5**

3.0 **3.1** 3.2 **3.3** 3.4 **3.5**

Mathematical Reasoning

1.0 1.1 1.2 1.3

2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7

3.0 3.1 3.2 3.3

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Chapter 3: Grade Six

Key Standards and Elaboration

NUMBER SENSE

Most of the standards in the Number Sense strand for the sixth grade are very important. These standards can be organized into four groups. The first is the comparison and ordering of positive and negative fractions (i.e., rational numbers), decimals, or mixed numbers and their placement on the number line:

1.1 Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.

The ordering of fractions is best done through the use of the *cross-multiplication algorithm*, which says $a/b = c/d$ exactly when $ad = bc$, and $a/b < c/d$ exactly when $ad < bc$. Students must not only be fluent in the use of this algorithm but also *understand why it is true*. The reason for the latter goes back to the observation already made in grades four and five that any two fractions can be rewritten as two fractions with the same denominator. Thus a/b and c/d can be rewritten as ad/bd and bc/bd. The cross- multiplication algorithm now becomes obvious.

Of particular importance is the students' understanding of the positions of the negative numbers and the geometric effect on the numbers of the number line when a number is added or subtracted from them.

The second group is represented by the next three standards, all of which refer to ratios and percents:

1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b, a to b, a:b).

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1.3 Use proportions to solve problems (e.g., determine the value of N if $4/7 = N/21$, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

1.4 Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

Notice that although Standards 1.2 and 1.3 come before Standard 2.1, they need to be taught after students know all about Standard 2.1; that is, after they have learned about the multiplication and division of fractions (for example, Standard 1.3 explicitly uses the language of “multiplicative inverse”). Once that is done, a *ratio* can then be defined as the division of one number by another; for example, the ratio of miles traveled to hours traveled (miles per hour), the ratio of the weights of two bags of potatoes, and so forth. In Standard 1.4 the teacher must be sure to explain why the concept of *percent* is useful: it standardizes the comparison of magnitudes and, in most situations, facilitates computations. For example, one can imagine the confusion that would arise if the sales tax of one state is 17/200 and that of another state is 4/45. Which state has a higher sales tax? By agreeing to express the tax as a percent, the two states would normalize their taxes to 8.5% and 8.9% (approx.), respectively. Then one can tell at a glance that the second one is higher. Of course, the expression in terms of percent makes the computation of sales tax relatively easy: an 8.5% tax on an article of \$25.50 is $25.50 \times 0.085 = \$2.17$.

The third group includes the remaining Number Sense standards, all of which relate to fractions:

2.0 Students calculate and solve problems involving addition, subtraction, multiplication, and division.

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Because of the slight ambiguity of the language in Standard 2.0, it should be made explicit that *this standard deals with the four arithmetic operations of positive fractions as well as positive and negative integers*. The arithmetic operations of the rational numbers, that is, positive and negative fractions, in full generality are left to grade seven.

Since the addition and subtraction of fractions have been taught in grade five (Number Sense Standard 2.3), the main emphasis of sub-Standards 2.1 and 2.2 is on the multiplication and division of positive fractions. A common mistake is to launch immediately into the formula $a/b \times c/d = ac/bd$ without first giving meaning to the product of fractions $a/b \times c/d$. One can define $a/b \times c/d$ as the area of a rectangle with side lengths a/b and c/d (in which case the whole of which the product measures a part is the area of the unit square), or as the fraction which is *a* parts of *c/d* when *c/d* is divided into *b* equal parts. Both interpretations are useful in problem solving, and should be clearly explained.

From the explanation of Grade 5 standard 2.4 (Number Sense) in this chapter, the division of fractions is now straightforward: the expression $a/b / c/d = m/n$ means the same thing as $a/b = m/n \times c/d$. Grade 4 standard 2.2 (Algebra and Functions), students know that the equation will hold if both sides are multiplied by d/c, and therefore $a/b \times d/c = m/n \times c/d \times d/c$. The product of the last two fractions is just 1, so $m/n = a/b \times d/c$, and the invert-and-multiply rule for division of fractions is shown to be valid.

Standard 2.1 calls for solving problems that make use of multiplication and division of fractions. It is important that students know why the invert-and-multiply rule is sufficient for these applications.

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It was mentioned in the discussion of grade five in this chapter that the concept of least common multiple plays a role in the teaching of fractions. The following standard makes this point explicit:

2.4 Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

The use of the lcm (least common multiple) in fractions should be carefully qualified. On the one hand, a knowledge of lcm does lead to simplifications in some situations, e.g., $\frac{3}{16}-\frac{1}{24}=\frac{(3\times 3)-(2-1)}{48}=\frac{7}{48}$, where we have made use of the lcm of 16 and 24 being 48. This is obviously simpler than using the denominator 16 $\times$ 24. On the other hand, finding the lcm of the denominators can be computationally intensive. For example, is it faster, when adding 2/57 + 3/95, to determine the lcm of the denominators (which is 285), or simply use their product as a common denominator? 2/57 $+$ 3/95 as

$$\frac{(2\times 95)+(3\times 57)}{57\times 95}=\frac{361}{57\times 95}=\frac{361}{5415}$$

Reducing 361/5415 to 1/15 may be more difficult than finding the lcm first, and then reducing 19/285 to the same, and so the decision on whether to use the lcm should be based on an estimate of the more straightforward method, and whether there is a need to generate a reduced form of the sum.

The fourth group stands alone because it consists of only one standard:

2.3 Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations, that use positive and negative integers and combinations of these operations.

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For the first time, students are asked to be completely fluent with the arithmetic of negative integers. Students find this difficult because the reasons for some of the more basic rules seem obscure to them. The addition of positive integers may not be an issue, but if one of a and b is negative in $a + b$, then how should a student evaluate this sum? The most important thing to remember is that for any integer a, $-a$ is the number satisfying $a + (-a) = 0$. We now see how to add two negative numbers,

$$(-3) + (-5) = -(3 + 5),$$

because the number $[(-3) + (-5)]$ satisfies $[(-3) + (-5)] + \{3+5\} = (-3) + 3 + (-5) + 5 = 0 + 0 = 0$ (where the associative and commutative laws were employed), so that $[(-3) + (-5)] + [3 + 5] = 0$, which means $[(-3) + (-5)] = -(3+5)$. In general, if a and b are positive integers, then

$$(-a) + (-b) = -(a + \mathrm{b}).$$

This is because $[(-\mathrm{a}) + (-\mathrm{b})] + (\mathrm{a} + \mathrm{b}) = (-\mathrm{a}) + \mathrm{a} + (-\mathrm{b}) + \mathrm{b} = 0 + 0 = 0$ (where again the associative and commutative laws were used), so that $[(-a) + (-b)] + (a + b) = 0$, which then implies $(-a) + (-b) = -(a + b)$. If a and b are positive integers and $a < b$, then $a + (-b)$ can be computed in the following way: let c be a positive integers so that $a + c = \mathrm{b}$, then

$$a + (-b) = -c.$$

Here is why. We have just seen that $-b = -(a + c) = (-a) + (-c)$ and so $a + (-b) = a + (-a) + (-c) = 0 + (-c) = -c$, as claimed. In like manner, we can show that if $a + c = b$ for positive integers a, b, c, then

$$(-a) + b = c,$$

because $(-a) + b = (-a) + a + c = c$. We have just showed how to add any two integers.

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Now for the multiplication of integers, we first observe that, say, $(-3) \times 5 = -(3 \times 5)$. It is sufficient to show, by the usual reasoning, that $[(-3) \times 5] + [3 \times 5] = 0$. This is so because we make use of the distributive law and obtain, $[(-3) \times 5] + [3 \times 5] = [(-3) + 3] \times 5 = 0 \times 5 = 0$. More generally, and by the same reasoning, if *a* and *b* are any two integers, then

$$(-a) \times b = -(a \times b).$$

It similarly follows that $(-a)\times(-b) = -\left(a\times(-b)\right) = -\left(-\left(a\times b\right)\right) = \left(-1\times-1\right)\times\left(a\times b\right)$.

It remains to be shown that

$$(-1) \times (-1) = 1.$$

From Grade 4 standard 2.1 (Algebra and Functions), it follows that if this equation is true, then $(-1)\times(-1)+(-1)=1+(-1)=0$, but by the distributive law, $[(-1) \times (-1)] + (-1) = [(-1) \times (-1)] + [(-1) \times 1] = (-1) \times [(-1) + 1] = (-1) \times 0 = 0$, which is then exactly what is to be proved. To sum up, we now know

$$(-a)\times(-b) = (-1\times-1)\times(a\times b) = 1\times(a\times b) = a\times b.$$

ALGEBRA AND FUNCTIONS

In the Algebra and Functions strand, the important standards are 1.1 and 2.2. The standard that follows is an expansion of the discussion of linear equations that was begun in the fifth grade:

1.1 Write and solve one-step linear equations in one variable.

Students in the sixth grade should understand and be able to solve simple one-variable equations which are critically important for all applied areas of mathematics. At a more advanced grade level, students will be required to solve systems of linear equations. In Grade 6 they should be able to justify each step in evaluating linear equations as cited in Standard 1.3 (Algebra and Functions). This skill is critical to the algebraic reasoning that is to follow and to the development of carefully applied logic at each step of the process.

Standard 1.1 is closely related to the standards for ratio and percent in the Number Sense strand (Standards 1.2 and 1.4).

2.2 Demonstrate an understanding that *rate* is a measure of one quantity per unit value of another quantity.

Standard 2.2 emphasizes the importance of *understanding* the meaning of the concepts of rate and ratio. Rate and ratio are merely different interpretations in different contexts of dividing one number by another. This standard is also closely related to the problems of rates, average speed, distance, and time that are introduced in Standard 2.3.

MEASUREMENT AND GEOMETRY

The following core standards are a part of the Measurement and Geometry strand:

1.1 Understand the concept of a constant such as π; know the formulas for the circumference and area of a circle.

2.2 Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.

One can define π in many different ways. The recommendation here is to define it as the ratio of the circumference to diameter. The latter is built on *two* concepts relatively new to students, *ratio* and *length of a curve* (circumference), whereas the former uses only the concept of area. Moreover, the area of the unit circle can be approximated directly by the use of (good) grid papers, and students have a good chance of getting $\pi = 3.14 \pm 0.05$. This would not only create a strong impression on students but also deepen their understanding of both the number π and the concept of area.

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Standard 1.3 is also important, and students should know that the volumes of three-dimensional figures can often be found by dividing and combining them into figures whose volumes are already known.

STATISTICS, DATA ANALYSIS, AND PROBABILITY

The study of statistics is more important in the sixth grade than in the earlier grades. One of the major objectives of studying this topic in the sixth grade is to give students some tools to help them understand the uses and misuses of statistics. The core standards for Statistics, Data Analysis, and Probability that focus on these goals are:

2.2 Identify different ways of selecting a sample (e.g., convenience sampling, responses to a survey, random sampling) and which method makes a sample more representative for a population.

2.3 Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.

2.4 Identify data that represent sampling errors and explain why the sample (and the display) might be biased.

2.5 Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.

For example, if a study of computer use is focused solely on students from Fresno, the class might try to determine how valid the conclusions might be for the students in the entire state. Again, how valid would the conclusion of a study that interviewed 23 teachers from all over the state be for all the teachers in the state? These questions represent major applications of the type of precise and critical thinking that mathematics is supposed to facilitate in students.

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In the sixth grade, students are also expected to become familiar with some of the more sophisticated aspects of probability. They start with the following standard:

3.1 Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.

This strand is challenging but vitally important, not only for its use in statistics and probability but also as an illustration of the power of attacking problems systematically.

The concepts in probability Standards 3.3 and 3.5 may be difficult for students to understand:

3.3 Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if *P* is the probability of an event, 1-*P* is the probability of an event not occurring.

3.5 Understand the difference between independent and dependent events.

The topics in both standards need to be carefully introduced, and the terms must be defined. Both the concept that probabilities are measures of the likelihood that events might occur (numerical values for probabilities are usually expressed as numbers between 0 and 1) and the distinction between dependent and independent events are important for students to understand. If students can grasp the meaning of the terms, they can understand the basic points of these standards. This knowledge can help students reach accurate conclusions about statistical data.

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Considerations for Grade-Level Accomplishments in Grade Six

At the beginning of grade six, students need to be assessed carefully on their knowledge of the core content taught in the early grades, which is described at the beginning of the section for grade five, and on the following content from grade five:

— Increased fluency with the long- division algorithm

— Conversion of percents, decimals, and fractions, including examples that represent a value over 1 (e.g., $2.75 = 2\frac{3}{4} = 275\%$)

– Use of exponents to show the multiples of a single factor

— Addition, subtraction, multiplication, and division with decimal numbers and negative numbers

— Addition of fractions with unlike denominators and multiplication and division of fractions

All of these topics require teaching over an extended period of time. A systematic program must be established so that students can reach high rates of accuracy and fluency with these skills.

All topics delineated in the grade six standards, and in particular the key strands, should be assessed regularly throughout the sixth grade. Once the skills have been taught and mastery demonstrated through assessment, teachers need to continue to review and maintain the students' skills. Mental mathematics, warm-up activities, and additional questions on tests can be used to accomplish this task.

Important mathematical skills and concepts for students in grade six to acquire are as follows:

- The least common multiple and the greatest common divisor**.** Students can become confused by the concepts of the least common multiple (LCM) and the greatest common divisor (GCD). The least common multiple of two numbers includes examples in which one multiple is in fact the least common multiple (e.g.,

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2 and 8; the LCM is 8); the least common multiple is the product of the two numbers (e.g., 4 and 5; the LCM is 20); and the least common multiple is a number that fits into neither of the two first categories (6 and 8; the LCM is 24). The teaching sequence should include examples of all three types. Finding the LCM becomes much more difficult with large numbers (e.g., finding the LCM of 36 and 48). One way to determine the answers is with prime factors, $36 = 2 \times 2 \times 3 \times 3$ and $48 = 2 \times 2 \times 2 \times 2 \times 3$. The LCM is $2 \times 2 \times 2 \times 2 \times 3 \times 3$, or 144. The process for finding the LCM can be confused with the process for finding the greatest common divisor (what is the GCD of 12 and 16?) because both deal with multiples of prime factors of numbers. Students should also be told that when a number is very large (e.g., 250 digits), finding its prime factorization is impractical, even with the help of the most powerful computers now available. There are other methods besides finding their prime factorization to determine the GCD and LCM.

- Discounts, interest, and tips**.** Within this realm are problems that range from simple one-step problems to more complex multistep problems. Programs must be organized so that easier problems are introduced first, followed by a thorough teaching of significantly more difficult problems. An example of a simple discount problem is, *A dress cost 50 dollars. There is a 10 percent discount. How many dollars will the discount be?* This problem is solved by performing the calculation for 10 percent of 50. If the problem asks, *How much will the dress cost with the discount?* the students would have to subtract the discount from the original price. A much more complex problem would be, *The sale price of a dress is 40 dollars. The discount was 20 percent. What was the original cost of the dress?* The problem might be solved through several procedures, all of which would involve the application of many more skills than those called for in the first problem. To work the third problem, the student has to know that the original price equates with

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100 percent and the sales price is 80 percent of the original price. One way of solving the problem is for the student to write the equation $0.80\ N = 40$, with N representing the original price. Thus $N = 40/0.80 = 50$. This way of solving the problem focuses on the increased emphasis on the use of variables in the Algebra and Functions strand. The computation skills needed to solve for N obviously need to be taught before this type of problem is introduced.

The treatment of interest at this grade is meant to deal with simple interest in one accrual period. It is not intended to extend to compound interest over several accrual periods in which the time is expressed as an exponent, as is the case for the normal computation formula for compound interest.

- Multiplication and division of fractions**.** Students should learn why and how fractions are multiplied and divided. Students must understand why the second fraction in a division problem is inverted, if that process is used. Students need to know when to use multiplication or division in application problems. For example, *There are 24 students in our class. Two-thirds of them passed the test. How many students passed the test?* is solved through multiplying; while the problem, *A piece of cloth that is 12 inches long is going to be cut into strips that are 2/3 of an inch long. How many strips can be made?* is solved through division. Structured systematic teaching must be done to help students determine which procedure to use in solving different problems.

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Chapter 3: Grade Seven Areas of Emphasis

By the end of grade seven, students are adept at manipulating numbers and equations and understand the general principles at work. Students understand and use factoring of numerators and denominators and properties of exponents. They know the Pythagorean theorem and solve problems in which they compute the length of an unknown side. Students know how to compute the surface area and volume of basic three-dimensional objects and understand how area and volume change with a change in scale. Students make conversions between different units of measurement. They know and use different representations of fractional numbers (fractions, decimals, and percents) and are proficient at changing from one to another. They increase their facility with ratio and proportion, compute percents of increase and decrease, and compute simple and compound interest. They graph linear functions and understand the idea of slope and its relation to ratio.

Number Sense

1.0 1.1 **1.2** 1.3 **1.4** **1.5** 1.6 **1.7**

2.0 2.1 **2.2** **2.3** 2.4 **2.5**

Algebra and Functions

1.0 1.1 1.2 **1.3** 1.4 1.5

2.0 2.1 2.2

3.0 3.1 3.2 **3.3** **3.4**

4.0 **4.1** **4.2**

Measurement and Geometry

1.0 1.1 1.2 **1.3**

2.0 2.1 2.2 2.3 2.4

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3.0 3.1 3.2 **3.3** **3.4** 3.5 **3.6**

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 **1.3**

Mathematical Reasoning

1.0 1.1 1.2 1.3

2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8

3.0 3.1 3.2 3.3

Chapter 3: Grade Seven

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Key Standards and Elaboration

NUMBER SENSE

The first basic standard for the Number Sense strand is:

1.2 Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.

At this point the students should understand arithmetic involving rational numbers. Negative fractions are formally introduced and studied for the first time. They should know the difference between rational and irrational numbers (Standard 1.4) and be aware that numbers such as the square root of two are not rational. Here, teachers should take care not to misinform the students. For example, some textbooks assert that the square root of 2 is not a rational number and then “prove” that assertion by

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producing a calculator-generated representation of $\sqrt{2}$ to perhaps 15 decimal places and state that the decimal is not repeating. That is unacceptable. It is better to use the facts in the standard (Standard 1.5) to construct an explicit nonrepeating decimal:

> **1.5** Know that every rational number is either a terminating or a repeating decimal and be able to convert terminating decimals into reduced fractions.

One can construct a nonrepeating decimal, for example, by putting zeros in all the places past the decimal point except for putting ones in (1) the first, second, fourth, and eighth places and generally the places marked by each power of 2:

$$0.110100010000000100000000000000010000\ .\ .\ .$$

or perhaps (2) the first, third, sixth, tenth, and generally, the places marked by $\frac{n(n+1)}{2}$:

$$0.101001000100001000001000000100\ .\ .\ .\ .$$

In this way students will see how to construct vast quantities of irrational numbers. At this point it might be possible to challenge the advanced students by showing them that a specific number (such as $\sqrt{2}$) is, in fact, irrational. They then can learn that while there are vast quantities of both rational and irrational numbers, it is often very difficult to show that specific numbers are in one set or the other. But this sophisticated material should not be emphasized for the class as a whole. In particular, at this stage it is probably not wise to attempt any kind of a proof of the facts in Standard 1.5. The students can be told that this basic awareness of irrationality is sufficiently important to be discussed at this point even though its justification will have to be deferred until they take a more advanced course.

By now the students should have enough skill with factoring integers so that they can use factoring to find the smallest common multiple of two whole numbers (Standard 2.2). Teachers should emphasize, once again, that the correct definition of

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the sum of two fractions is $(a/b) + (c/d) = (ad + bc)/bd$ and that the usual algorithm using factoring to find the smallest common denominator is but a refinement of the primary definition. (See the discussion in the Number Sense standards for the fifth grade.) For this topic students should become more familiar with the basic exponent rules (Standard 2.3), which will have direct applications in the main seventh grade application of compound interest.

The last topic in the first standard of the Number Sense strand (Standard 1.7) is also one of the high points of the entire strand:

1.7 Solve problems that involve discounts, markups, commissions, and profit and compute simple and compound interest.

This is a major topic, which should come toward the end of the year and should be a major highlight of the kindergarten through grade seven mathematical experience. It provides one of the most important applications of mathematics in students' everyday life, a skill that can mean the difference between students managing their money and other resources well or not at all.

Standard 2.5, the last standard in the Number Sense strand, on absolute value should receive some emphasis. This topic is usually slighted in middle schools and high schools; however, students should acquire some facility with this concept as early as possible. The students need to understand that the correct way to express the statement "two numbers x and y are close to each other" is "$|x-y|$ is small." The concept of two numbers being "close" was introduced in grade four in connection with rounding off (see "Elaboration" in grade four).

ALGEBRA AND FUNCTIONS

Familiarity with the distributive law, the associative law, and the commutative rule for addition and multiplication of whole numbers has been mentioned at several

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points previously in the Algebra and Functions standards in grades five and six. For these standards in grade seven, the concepts are taken a step further with the following:

1.3 Simplify numerical expressions by applying the properties of rational numbers (e.g., identity, inverse, distributive, associative, commutative) and justify the process used.

This is a critical step in learning how to abstract and shows the power of abstract thinking in helping to make sense of complex situations and derive their basic properties.

One of the most basic topics in applications of mathematics is systems of linear equations. A clear understanding of even something as simple as systems of two linear equations in two unknowns is crucial to understanding more advanced topics, such as calculus and analysis. The first major steps are taken toward this goal when the study of a single linear equation is initiated in these four standards:

3.3 Graph linear functions, noting that the vertical change (change in *y*-value) per unit of horizontal change (change in *x*-value) is always the same and know that the ratio ("rise over run") is called the slope of a graph.

3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the ratio of the quantities.

4.1 Solve two-step linear equations and inequalities in one variable over the rational numbers, interpret the solution or solutions in the context from which they arose, and verify the reasonableness of the results.

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4.2 Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

Again, the connection of the second standard with the Measurement and Geometry Standard 1.3 should be noted. These topics provide excellent problems to test the students' understanding of the techniques for solving linear equations.

Students at this stage of algebraic development should be able to understand a clarification of the somewhat subtle concepts of *ratio* and *direct proportion* (sometimes called *direct variation*). The "ratio between two quantities" is nothing more or less than a particular interpretation of "one quantity divided by another in the sense of numbers." Of course, thus far students know only how to divide rational numbers. The teacher should tell the students that the division between irrational numbers will also be explained to them in more advanced courses; therefore, this definition of *ratio* will still apply. *Direct variation* can be explained in terms of linear functions: "A varies directly with B" means that "for a fixed constant c, $A = cB$." Teachers and textbooks commonly try to "explain" the meanings of both terms in abstruse language, resulting in confusion among students and even teachers. No explanation is necessary: *ratio* and *direct variation* are mathematical terms, and they should be clearly defined once the students have been taught the necessary facts and techniques.

MEASUREMENT AND GEOMETRY

The first major emphasis in the Measurement and Geometry strand is for the students to develop an increased sense of spatial relations. This topic is reflected in these two standards:

3.4 Demonstrate an understanding of conditions that indicate two geometrical figures are congruent and what congruence means about the relationships between the sides and angles of the two figures.

3.6 Identify elements of three-dimensional geometric objects (e.g., diagonals of rectangular solids) and describe how two or more objects are related in space (e.g., skew lines, the possible ways three planes might intersect).

A critical part of understanding this material is that the students know the general definition of *congruence*—two figures are congruent if a succession of reflections, rotations, and translations will make one coincide with the other—and understand that properties of congruent figures, such as angles, edge lengths, areas, and volumes, are equal.

The next basic step is contained in the following standard:

3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

The Pythagorean theorem is probably the first true theorem that the students will have seen. It should be emphasized that students are not expected to prove this result. But the better students should be able to understand the proof given by cutting, in two different ways, a square with the edges of length $a + b$ (where a and b are the lengths of the legs of the right triangle). However, everyone is expected to understand what the theorem and its converse mean and how to use both. The applications can include understanding the formula that the square root of $x^2 + y^2$ is the length of the line segment from the origin to the point (x, y) in the plane and that

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the shortest distance from a point to a line not containing the point is the length of the line segment from the point perpendicular to the line.

Although the following topics are not as basic as the preceding ones, they should also be covered carefully. Seventh grade students should memorize the formulas for the volumes of cylinders and prisms (Standard 2.1). Students at this point should understand the discussion that began in the sixth grade concerning the volume of "generalized cylinders." More precisely, they should think of a right circular cylinder as the solid traced by a circular disc as this disc moves up a line segment *L* perpendicular to the disc itself. The disc is replaced with a planar region of any shape, and the line segment *L* is no longer required to be perpendicular to the planar region. Then, as the planar region moves up along *L,* always parallel to itself, it traces out a solid called a generalized cylinder. The formula for the volume of such a solid is still (height of the generalized cylinder) x (area of the planar region). *Height* now refers to the vertical distance between the top and bottom of the generalized cylinder.

The final topic to be emphasized in seventh grade Measurement and Geometry is as follows:

1.3 Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

This standard interacts well with the demands of the algebra standards, particularly in solving linear equations. Typically, the main difficulty in understanding problems of this kind is keeping the definitions and the physical significance of the various measures straight; therefore, care should be taken to emphasize the meanings of the terms in the various problems.

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STATISTICS, DATA ANALYSIS, AND PROBABILITY

The most important of the three seventh grade standards in Statistics, Data Analysis, and Probability is this:

1.3 Understand the meaning of, and be able to compute, the minimum, the lower quartile, the median, the upper quartile, and the maximum of a data set.

These are useful measures that students need to know well. Care should be taken to ensure that all students know the definitions, and many examples should be given to illustrate them.